POSITIVITY OF CONTINUOUS PIECEWISE POLYNOMIALS

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Abstract. Real algebraic geometry provides certificates for the positivity of polynomials on semi-algebraic sets by expressing them as a suitable combination of sums of squares and the defining inequalitites. We show how Putinar's theorem for strictly positive polynomials on compact sets can be applied in the case of strictly positive piecewise polynomials on a simplicial complex. In the 1-dimensional case, we improve this result to cover all non-negative piecewise polynomials and give explicit degree bounds.

Introduction

Let $\Delta = \sigma_1 \cup \cdots \cup \sigma_k$ be a simplicial complex in $\mathbb{R}^n$ with vertices $v_1, \ldots, v_m$. Let $\mathcal{C}^0(\Delta)$ denote the algebra of all continuous piecewise polynomials on Δ, consisting of all continuous functions $F\colon \Delta \to \mathbb{R}$ such that the restriction of F to each σ_i is given by a polynomial. It has been studied in connection with splines, where the functions are also required to be differentiable to some order. For a good survey, see Strang [13] and references given there. A detailed analysis of $\mathcal{C}^0(\Delta)$ from the point of view of combinatorics and commutative algebra is due to Billera [1].

The algebra $\mathcal{C}^0(\Delta)$ has a beautiful description by generators and relations, given in terms of its *tent functions* or *Courant functions*: These are the unique piecewise linear functions $T_i\colon \Delta \to \mathbb{R}$ such that $T_i(v_i) = 1$ and $T_i(v_j) = 0$ if $i \neq j$. The tent functions generate $\mathcal{C}^0(\Delta)$ and satisfy certain obvious relations, which Billera has shown to be sufficient to completely describe $\mathcal{C}^0(\Delta)$ (see Thm. 1.2 below). The relations are in fact identical to those of the *Stanley-Reisner ring* (or face ring) of Δ, plus one additional relation that accounts for the fact that the tent functions sum to 1 (see [1], [5, Def. 1.6]).

In this paper, we show how the tent functions can also be used to characterize positive and non-negative functions in $\mathcal{C}^0(\Delta)$. In general, much work in real algebraic geometry has been concerned with so-called certificates for positivity: Let $h_1, \ldots, h_r \in \mathbb{R}[t]$ be real polynomials in n variables $t = (t_1, \ldots, t_n)$, and let S be the basic closed semi-algebraic set $\{x \in \mathbb{R}^n \mid h_1(x) \geqslant 0, \ldots, h_r(x) \geqslant 0\}$. The convex cone $M \subset \mathbb{R}[t]$ generated by all polynomials g^2 and $g^2 h_i$ is called a *quadratic module.* Putinar [8] has shown that M contains all polynomials that are strictly positive on S if there exists $N \in \mathbb{Z}_+$ such that $N - \sum t_i^2 \in M$, in which case M is called *archimedean.* If M is archimedean, then S is clearly compact. Schmüdgen's positivstellensatz [11] (which predates that of Putinar) can be rephrased as saying that the quadratic module generated by all square-free products of $h_1, \ldots, h_r$ (called the preordering) is archimedean whenever S is compact. These and many related results have attracted attention in optimization because membership of a polynomial in a preordering or quadratic module can be checked (in practice rather efficiently) by a semidefinite program. Good general references on the subject are the books of Marshall [4] and Prestel and Delzell [7].

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A simplicial complex Δ is, of course, a special kind of semi-algebraic set, and the algebra $\mathcal{C}^0(\Delta)$ can be interpreted as the ring of polynomial functions on an affine algebraic variety containing Δ (which is just the Zariski-closure of Δ if the ambient dimension n is large and the vertices of Δ are in general position; see Prop. 1.1). It is therefore possible to translate known results into this setup, in particular Putinar's positivstellensatz (Thm. 2.1):

Theorem. *Let $\Delta \subset \mathbb{R}^n$ be a simplicial complex with m vertices. Let $T_1, \ldots, T_m \in \mathcal{C}^0(\Delta)$ be the tent functions on Δ. If a function $F \in \mathcal{C}^0(\Delta)$ is strictly positive on Δ, then there exist sums of squares S_i in $\mathcal{C}^0(\Delta)$, $i = 0, \ldots, m$, such that*

$$F = S_0 + \sum_{i=1}^{m} S_i T_i.$$

Beyond pointing out this application of a known result, the contribution of this paper is a strengthening in the case when Δ is 1-dimensional:

Theorem. *Let $\Delta \subset \mathbb{R}^n$ be a simplicial complex of dimension 1 with e edges and m vertices, of which m_0 are isolated. Let $T_1, \ldots, T_m \in \mathcal{C}^0(\Delta)$ be the tent functions on Δ. A function $F \in \mathcal{C}^0(\Delta)$ is non-negative on Δ if and only if there exist sums of squares S, S_{ij} in $\mathcal{C}^0(\Delta)$ such that*

$$F = S + \sum_{(i,j) \in E} S_{ij} T_i T_j.$$

More precisely, there exist such S and S_{ij} of degree at most $\deg(F) + 6(e-1) + 1$, where S is a sum of at most $2e + m_0$ squares and each S_{ij} is a sum of two squares.

Positive polynomials on general semialgebraic subsets of real algebraic curves have been studied extensively by Scheiderer in [9]. The criteria developed there cover all semialgebraic subsets of irreducible curves. The proof of the main result above given here is elementary and (in principle) constructive. Alternatively, it is possible to obtain a more abstract proof (without the degree bounds) using the local results and local-global principle in [9]. The general case of semialgebraic subsets of a reducible curve is not settled completely. Most of the results obtained by the author in [6] only apply to sums of squares. The existence of degree bounds for quadratic modules (*stability*) is also open in most 1-dimensional cases, even for irreducible curves (except when the curve is rational or elliptic). The case of a simplicial complex is thus a very particular one, not only because the curves involved are just lines, but also because the generators $T_i T_j$ of the quadratic module vanish in the intersection points (the vertices), which turns out to be very helpful.

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1. Preliminaries

Let $v_0, \ldots, v_d \in \mathbb{R}^n$ be affinely independent vectors, $d \leqslant n$. We will write $\sigma(v_0, \ldots, v_d)$ for the d-simplex spanned by $v_0, \ldots, v_d$, which is the convex hull of $\{v_0, \ldots, v_d\}$. It is equivalent to the d-dimensional standard simplex $\sigma(e_1, \ldots, e_{d+1})$ up to an affine change of coordinates. The faces of $\sigma(v_0, \ldots, v_d)$ are precisely the 2^{d+1} subsimplices $\sigma(v_{i_0}, \ldots, v_{i_e})$ with $\{i_0, \ldots, i_e\} \subset \{0, \ldots, d\}$, $e \leqslant d$.

A *simplicial complex* is a union $\Delta = \sigma_1 \cup \cdots \cup \sigma_k$, where each σ_i is a simplex in $\mathbb{R}^n$, such that $\sigma_i \cap \sigma_j$ is a face of both σ_i and σ_j, for all $i, j = 1, \ldots, k$. One can always assume that $\sigma_i \not\subseteq \sigma_j$ for all i, j. In that case, Δ is called *pure dimesional* if all σ_i have the same dimension.

Let $d_i = \dim(\sigma_i)$, and write $\sigma_i = \sigma(v_{i0}, \ldots, v_{id_i})$. We denote by $V_\Delta = \{v_{ij} \mid i = 1, \ldots, k, j = 0, \ldots, d_i\}$ the set of all vertices of Δ. The simplicial complex Δ is uniquely

determined by the set V_Δ together with the information which of the simplices $\sigma(V')$ for $V' \subset V_\Delta$ are contained in Δ.

A function $f\colon \Delta \to \mathbb{R}$ will be called a *piecewise polynomial on* Δ if there exist polynomials $g_1, \dots, g_k \in \mathbb{R}[t_1, \dots, t_n]$ such that $f|_{\sigma_i} = g_i|_{\sigma_i}$, for all $i = 1, \dots, k$. We will study positivity in the algebra

$$\mathcal{C}^0(\Delta) = \big\{ f\colon \Delta \to \mathbb{R} \;\big|\; f \text{ is continuous and piecewise polynomial} \big\}.$$

As explained in the introduction, we will work with two different descriptions of $\mathcal{C}^0(\Delta)$, the first of which is given as the coordinate ring of an affine $\mathbb{R}$-variety. For the purpose of this paper, such a variety is given by an $\mathbb{R}$-algebra $\mathbb{R}[U]$ that is reduced (i.e. without nilpotents) but not necessarily integral. The geometric object U associated with $\mathbb{R}[U]$ is the set $\mathrm{Hom}(\mathbb{R}[U], \mathbb{R})$ of $\mathbb{R}$-algebra homomorphisms. After fixing coordinates, i.e. a surjection $\varphi\colon \mathbb{R}[t_1, \dots, t_N] \to \mathbb{R}[U]$ for some N and a finite set of generators $f_1, \dots, f_r$ of $\ker(\varphi)$, the set U is in canonical bijection with $\{x \in \mathbb{R}^N \mid f_1(x) = \cdots f_r(x) = 0\}$, the common zero-set of $f_1, \dots, f_r$. Note that we only consider real points here (i.e. we identify U with $U(\mathbb{R})$ in the notation of [9], [6]), which is sufficient for our needs.

Proposition 1.1. *Let $\Delta = \sigma_1 \cup \cdots \cup \sigma_k \subset \mathbb{R}^n$ be a simplicial complex. For all $i \leqslant j = 1, \dots, k$, let U_{ij} be the affine hull of $\sigma_i \cap \sigma_j$. Let U be the affine $\mathbb{R}$-variety obtained as the direct limit of the directed system $\{U_{ij}\}$ ordered by the inclusions $U_{ij} \to U_{ii}$, $U_{ij} \to U_{jj}$. Then*

$$\mathcal{C}^0(\Delta) \cong \mathbb{R}[U] \cong \left\{ (f_1, \dots, f_k) \in \prod_{i=1}^k \mathbb{R}[U_{ii}] \;\Big|\; f_i|_{U_{ij}} = f_j|_{U_{ij}} \textit{ for all } i \neq j \right\}.$$

Proof. It suffices to note that the values of a polynomial g_i on $\sigma_i \cap \sigma_j$ uniquely determine its values on the Zariski-closure, which is U_{ij}. □

If all vertices of Δ are in sufficiently general position, so that the affine spans U_{ii} of σ_i are all distinct subspaces of $\mathbb{R}^n$, then U is isomorphic to the union $\bigcup_{i=1}^k U_{ii}$. One can always arrive at this situation by embedding Δ into a higher-dimensional ambient space, if necessary.

Once we fix isomorphisms $U_{ii} \cong \mathbb{R}^{d_i}$ (even though there is no canonical choice), this fixes U_{ij} as affine subspaces of U_{ii}, U_{jj}, and we can think of the elements of $\mathcal{C}^0(\Delta)$ as k-tuples of polynomials together with a compatibility condition:

$$\mathcal{C}^0(\Delta) \cong \big\{ (g_1, \dots, g_k) \in \prod_{i=1}^k R[t_1, \dots, t_{d_i}] \;\big|\; g_i|_{U_{ij}} = g_j|_{U_{ij}} \text{ for all } i, j = 1, \dots, k \big\}.$$

The second, more intrinsic description of $\mathcal{C}^0(\Delta)$ in terms of its tent functions, was given by Billera in [1]: Let $V_\Delta = \{v_1, \dots, v_m\}$ be the set of vertices in Δ, as above. For $i = 1, \dots, m$, let T_i be the *tent function* or *Courant function* on Δ, which is the unique piecewise linear function in $\mathcal{C}^0(\Delta)$ with $T_i(v_i) = 1$ and $T_i(v_j) = 0$ for $i \neq j$.

Theorem 1.2 (Billera [1], Thm. 2.3 and Thm. 3.6)**.** *The tent functions $T_1, \dots, T_m$ generate $\mathcal{C}^0(\Delta)$. The kernel of the map $\mathbb{R}[X_1, \dots, X_m] \to \mathcal{C}^0(\Delta)$ given by $X_i \mapsto T_i$ is generated by the elements*

- $1 - \sum_{i=1}^m X_i$
- $X_{i_1} \cdots X_{i_e}$ *whenever $\{i_1, \dots, i_e\} \subset \{1, \dots, m\}$ is such that $\Delta(v_{i_1}, \dots, v_{i_e})$ is not contained in Δ.*

Remark 1.3. Billera makes the additional assumption that Δ be pure-dimensional. However, this appears to be immaterial for that particular part of his paper.

We will mostly be concerned with the case when Δ is of dimension 1. We will write $E_\Delta = \{(i,j) \in \{1, \dots, n\}^2 | i < j \text{ and } \Delta(v_i, v_j) \in \Delta\}$ for the set of indices corresponding to

the edges of Δ. By Billera's theorem, $\mathcal{C}^0(\Delta)$ is generated by the tent functions $T_1,\ldots,T_m$ subject to the rules:

(1) $\sum_{i=1}^m T_i = 1$.
(2) $T_iT_jT_k = 0$ for all distinct i,j,k.
(3) $T_iT_j = 0$ if and only if $(i,j)\notin E_\Delta$.

We will have to go back and forth between the two descriptions of $\mathcal{C}^0(\Delta)$ that we have seen. Our description of the affine variety associated with Δ in Prop. 1.1, together with explicit coordinates, translates into the following proposition, the proof of which is obvious:

Proposition 1.4. *Let $\Delta\subset\mathbb{R}^n$ be a purely 1-dimensional simplicial complex with vertices $V_\Delta=\{v_1,\ldots,v_m\}$. For every edge $(i,j)\in E_\Delta$, let C_{ij} be a copy of $\mathbb{R}$ with coordinate ring $\mathbb{R}[t_{ij}]$. Let φ be the unique map from the disjoint union of all C_{ij}, $(i,j)\in E_\Delta$, into $\mathbb{R}^n$ taking C_{ij} to the line $v_i+\mathbb{R}\cdot(v_i-v_j)$ and mapping $-1\in C_{ij}$ to v_i and $1\in C_{ij}$ to v_j. The dual ring homomorphism $\varphi^*\colon\mathcal{C}^0(\Delta)\to\prod_{(i,j)\in E_\Delta}\mathbb{R}[t_{ij}]$ induces an isomorphism*

$$\mathcal{C}^0(\Delta)\cong\left\{(f_{ij})_{(i,j)\in E_\Delta}\;\middle|\;\begin{array}{ll} f_{ij}(-1)=f_{ik}(-1) & \text{if } (i,j),(i,k)\in E_\Delta\\ f_{ij}(1)=f_{jk}(-1) & \text{if } (i,j),(j,k)\in E_\Delta\\ f_{ij}(1)=f_{kj}(1) & \text{if } (i,j),(k,j)\in E_\Delta\end{array};i,j,k=1,\ldots,m\right\}$$

Under this isomorphism, the tent function T_k corresponds to the function $(f_{ij})_{(i,j)\in E_\Delta}$ with $f_{ik}=\frac{1}{2}(1+t_{ik})$ and $f_{kl}=\frac{1}{2}(1-t_{kl})$ for all $(i,k),(k,l)\in E_\Delta$ and $f_{ij}=0$ for all $(i,j)\in E_\Delta$ with $i,j\neq k$.
In particular, the product T_iT_j with $(i,j)\in E_\Delta$ is $(0,\ldots,\frac{1}{4}(1-t_{ij}^2),\ldots,0)$. □

We have to say what the degree of a piecewise polynomial should be:

Definition 1.5. Let $\Delta=\sigma_1\cup\cdots\cup\sigma_k\subset\mathbb{R}^n$. Given $F\in\mathcal{C}^0(\Delta)$, let $\mathcal{R}(F)$ be the set of all k-tuples of polynomials $f_i\in\mathbb{R}[t_1,\ldots,t_n]$ with $F|_{\sigma_i}=f_i|_{\sigma_i}$. Define the *degree* of F, denoted $\deg(F)$, as $\deg(F)=\min_{(f_i)\in\mathcal{R}(F)}\{\max_i\{\deg(f_i)\}$.

Remark 1.6. If we identify $\mathcal{C}^0(\Delta)$ with $\{(g_1,\ldots,g_k)\in\prod_{i=1}^k R[t_1,\ldots,t_{d_i}]\mid g_i|_{U_{ij}}=g_j|_{U_{ij}}\}$ as in Prop. 1.1, then F has a unique representation $F=(g_1,\ldots,g_k)$ with $g_i\in\mathbb{R}[t_1,\ldots,t_{d_i}]$, and $\deg(F)=\max_i\{\deg(g_i)\}$.

On the other hand, every $F\in\mathcal{C}^0(\Delta)$ can be expressed (non-uniquely) as a polynomial in the tent functions $T_1,\ldots,T_m$. Let $\mathcal{T}(F)=\{G\in\mathbb{R}[t_1,\ldots,t_m]\mid F=G(T_1,\ldots,T_m)\}$ and define $\deg_{\mathcal{T}}(F)=\min\{\deg(G)\mid G\in\mathcal{T}(F)\}$. Since $\deg(T_i)=1$, we have $\deg(F)\leqslant\deg_{\mathcal{T}}(F)$. In general, this inequality may be strict. For example, if Δ consists of two isolated points, then $\deg(F)=0$ for all $F\in\mathcal{C}^0(\Delta)$ but $\deg_{\mathcal{T}}(F)=1$ whenever F is non-constant.

Remark 1.7. Write $\mathcal{C}^0(\Delta)$ as in Prop. 1.4 and fix an edge $(k,l)\in E_\Delta$. Then given any $g\in\mathbb{R}[t_{kl}]$, there exists $(f_{ij})\in\mathcal{C}^0(\Delta)$ with $f_{kl}=g$ and $\deg(f_{ij})\leqslant 1$ for all $(i,j)\neq(k,l)$. In fact, we can take $f_{ij}=0$ for $\{i,j\}\cap\{k,l\}=\emptyset$, and if $(k,i)\in E$ for some $i\neq l$, put $f_{ki}=\frac{g(-1)}{2}(1-t_{ki})$, and similarly in the remaining cases.

Finally, we set up some notation and terminology for quadratic modules: Let A be a ring (commutative with unit). By $\sum A^2$, we denote the set of all sums of squares of elements in A. For finitely many elements $h_1,\ldots,h_r\in A$, we write

$$\mathrm{QM}(h_1,\ldots,h_r)=\left\{s_0+\sum_{i=1}^r s_ih_i\mid s_0,\ldots,s_r\in\sum A^2\right\}$$

and call this the *quadratic module* generated by $h_1,\ldots,h_r$. The quadratic module $M=\mathrm{QM}(h_1,\ldots,h_r)$ is called *archimedean* if for every $f\in A$ there exists $N\in\mathbb{Z}_+$ such that $N+f\in M$.

Proposition 1.8. [4, Cor. 5.2.4] *With A and M as above, assume that A is finitely generated over a field by elements $t_1, \dots, t_m$. The following are equivalent:*

(1) M is archimedean.
(2) There exists $N \in \mathbb{Z}_+$ such that $N - \sum_{i=1}^m t_i^2 \in M$.
(3) There exists $N \in \mathbb{Z}_+$ such that $N \pm t_i \in M$ for all $i = 1, \dots, m$.

2. Positivity in $\mathcal{C}^0(\Delta)$

Recall from the introduction the statement of Putinar's positivstellensatz:

Theorem 2.1 (Putinar [8])**.** *Let U be an affine $\mathbb{R}$-variety with coordinate ring $\mathbb{R}[U]$, let $h_1, \dots, h_r \in \mathbb{R}[U]$ and $K = \{x \in U \mid h_1(x) \geqslant 0, \dots, h_r(x) \geqslant 0\}$. If the quadratic module $\mathrm{QM}(h_1, \dots, h_r)$ is archimedean, then it contains every $f \in \mathbb{R}[U]$ such that $f(x) > 0$ holds for all $x \in K$.*

In the original paper, as well as in [7], the theorem is stated for $U = \mathbb{R}^n$. But it is straightforward to pass to the version given here: Fix a surjection $\varphi\colon \mathbb{R}[t_1, \dots, t_N] \to \mathbb{R}[U]$ and a finite set of generators $G_1, \dots, G_s$ of $\ker(\varphi)$, giving an embedding $U = \{x \in \mathbb{R}^N \mid G_1(x) = \cdots = G_s(x) = 0\}$. Choose $H_1, \dots, H_r \in \mathbb{R}[t_1, \dots, t_N]$ such that $\varphi(H_i) = h_i$. Then $K = \{x \in \mathbb{R}^N \mid H_i(x) \geqslant 0, G_j(x) \geqslant 0, -G_j(x) \geqslant 0 \text{ for all } i, j\}$. Put $M = \mathrm{QM}(h_1, \dots, h_r) \subset \mathbb{R}[U]$ and $M_0 = \mathrm{QM}(H_1, \dots, H_r, \pm G_1, \dots, \pm G_s) \subset R[t_1, \dots, t_n]$, so that $\varphi(M_0) = M$. That M is archimedean means that there exists $N \in \mathbb{Z}_+$ such that $N - \sum \varphi(t_i)^2 \in M$. From this we conclude $N - \sum t_i^2 \in M_0$, so that M_0 is archimedean, too. Given $f \in \mathbb{R}[U]$ as in the theorem, we may choose $F \in \mathbb{R}[t_1, \dots, t_N]$ with $\varphi(F) = f$ and conclude $F \in M_0$. Applying φ gives the desired representation of f in M. Alternatively, one can deduce the above version of Putinar's result directly from Jacobi's more general representation theorem (see [2] or [4, Thm. 5.4.4]).

Corollary 2.2. *Let $\Delta \subset \mathbb{R}^n$ be a simplicial complex with m vertices. Let $T_1, \dots, T_m \in \mathcal{C}^0(\Delta)$ be the tent functions on Δ. If a function $F \in \mathcal{C}^0(\Delta)$ is strictly positive on Δ, then there exist sums of squares S_i in $\mathcal{C}^0(\Delta)$, $i = 0, \dots, m$, such that*

$$F = S_0 + \sum_{i=1}^m S_i T_i$$

Proof. Let U be the affine variety associated with Δ, as defined in Prop. 1.1. Then it is easy to check that the tent functions define Δ as a subset of U, i.e. $\Delta = \{x \in U(\mathbb{R}) \mid T_1(x) \geqslant 0, \dots, T_m(x) \geqslant 0\}$. The quadratic module $\mathrm{QM}(T_1, \dots, T_m)$ is archimedean by Prop. 1.8, since it contains $1 - T_i = \sum_{j \neq i} T_j$ for all $i = 1, \dots, m$. □

Remark 2.3. Degree bounds for the sums of squares S_i that depend only on the degree of F cannot exist as soon as the dimension of Δ is at least two (see Scheiderer [10]). However, there exist bounds that depend on other data, in particular the minimum of F on Δ (see Schweighofer [12]).

The following is a special case of the general results of Kuhlmann, Marshall, and Schwartz for subsets of the line (see [3], §4).

Theorem 2.4. *Every $f \in \mathbb{R}[t]$ such that $f|_{[-1,1]} \geqslant 0$ admits a representation*

$$f = s_0 + s_1(1 - t^2)$$

where s_0, s_1 are sums of two squares with $\deg(s_0) \leqslant \deg(f) + 1$, $\deg(s_1) \leqslant \deg(f) - 1$.

Proof. Let $f \in \mathbb{R}[t]$ with $f|_{[-1,1]} \geqslant 0$. By [3, Thm. 4.1], there is a representation $f = r_0 + r_1(1+t) + r_2(1-t) + r_3(1-t^2)$ with the degree of each summand bounded by $\deg(f)$. Now substitute the identity $(1 \pm t) = \frac{1}{2}(1 \pm t)^2 + \frac{1}{2}(1 - t^2)$. □

Translated into our setup, this says:

Corollary 2.5. *If Δ is the 1-simplex with tent functions T_1, T_2, then a function $F \in \mathcal{C}^0(\Delta)$ is non-negative on Δ if and only if there exist sums of two squares S, S_{12} in $\mathcal{C}^0(\Delta)$ of degree at most $\deg(F)+1$ such that*

$$F = S + S_{12}T_1T_2$$

□

Our main result is a generalization to 1-dimensional simplicial complexes, which we restate from the introduction.

Theorem 2.6. *Let $\Delta \subset \mathbb{R}^n$ be a simplicial complex of dimension 1 with e edges and m vertices, of which m_0 are isolated. Let $T_1, \dots, T_m \in \mathcal{C}^0(\Delta)$ be the tent functions on Δ. A function $F \in \mathcal{C}^0(\Delta)$ is non-negative on Δ if and only if there exist sums of squares S, S_{ij} in $\mathcal{C}^0(\Delta)$ such that*

$$F = S + \sum_{(i,j)\in E} S_{ij}T_iT_j.$$

More precisely, there exist such S and S_{ij} of degree at most $\deg(F) + 6(e-1) + 1$, where S is a sum of at most $2e + m_0$ squares and each S_{ij} is a sum of two squares.

Remark 2.7. The quadratic module $\mathrm{QM}(T_iT_j \,|\, (i,j) \in E)$ used in the theorem coincides in fact with the quadratic module $\mathrm{QM}(T_1, \dots, T_m)$ used earlier. This follows from the identities $T_i = T_i(\sum_{j=1}^m T_j) = T_i^2 + \sum_{(i,j)\in E} T_iT_j$ and $T_iT_j = T_iT_j(\sum_{k=1}^m T_k) = T_iT_j(T_i + T_j) = T_i^2T_j + T_j^2T_i$, which was pointed out to me by the referee. In particular, $\mathrm{QM}(T_1, \dots, T_m)$ is in fact a preordering, i.e. it is closed under multiplication. Using these identities, one could also restate the degree bounds in Thm. 2.6 for $\mathrm{QM}(T_1, \dots, T_m)$.

I would like to thank Claus Scheiderer for suggesting the proof of the following lemma, replacing a much more pedestrian argument in an earlier version:

Lemma 2.8. *Let $f \in \mathbb{R}[t]$ be such that $f(x) \geqslant 0$ for all $x \in [-1,1]$. For every $a, b \in \mathbb{R}$ with $a^2 \leqslant f(-1)$ and $b^2 \leqslant f(1)$, there exists $s \in \mathbb{R}[t]$ with $\deg(s^2) \leqslant \deg(f) + 3$ such that $s(-1) = a$, $s(1) = b$ and such that $s^2(x) \leqslant f(x)$ for all $x \in [-1,-1]$.*

Proof. By Thm. 2.4, there exist sums of squares $s_0, s_1 \in \mathbb{R}[t]$ such that $f = s_0 + s_1(1-t^2)$ and with s_0 of degree $2d \leqslant \deg(f) + 1$. Factor $s_0 = \prod_{i=1}^d (t-\lambda_i)(t-\overline{\lambda_i})$ over $\mathbb{C}$ and let $g = \prod_{i=1}^d (t-\lambda_i)$. It follows that $g(-1) = \sqrt{f(-1)}\alpha$, $g(1) = \sqrt{f(1)}\beta$, where $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$. Put

$$\ell(t) = \frac{\overline{\alpha}a(1-t)}{2\sqrt{f(-1)}} + \frac{\overline{\beta}b(1+t)}{2\sqrt{f(1)}}$$

and let $s = \mathrm{Re}(g \cdot \ell) \in \mathbb{R}[t]$ (where the real part is taken on coefficients). By the choice of $\ell(t)$, we find that $s(-1) = a$ and $s(1) = b$. Furthermore, since $|\ell(x)| \leqslant 1$ for all $x \in [-1,1]$, we obtain

$$s^2(x) = \big(\mathrm{Re}(g(x)\cdot\ell(x))\big)^2 \leqslant |g(x)\cdot\ell(x)|^2 \leqslant |g(x)|^2 = s_0(x) \leqslant f(x)$$

for all $x \in [-1,1]$. □

Corollary 2.9. *Let $f \in \mathbb{R}[t]$ be such that $f(x) \geqslant 0$ for all $x \in [-1,1]$. Given $k \in \mathbb{Z}_+$ and vectors $a, b, \in \mathbb{R}^k$ such that $f(-1) = ||a||$ and $f(1) = ||b||$, there exist polynomials $s_1, \dots, s_{k+2} \in \mathbb{R}[t]$ and $r \in \sum \mathbb{R}[t]^2$ such that the following hold:*

1. *$f = \sum_{i=1}^{k+2} s_i^2 + r(1-t^2)$.*
2. *$s_i(-1) = a_i$, $s_i(1) = b_i$ for all $i = 1, \dots, k$*
3. *$s_i(-1) = 0$, $s_i(1) = 0$ for $i = k+1, k+2$*
4. *$\deg(r), \deg(s_i^2) \leqslant \deg(f) + 3k + 1$ for all $1 \leqslant i \leqslant k+2$.*

Proof. Let $d = \deg(f)$. By the lemma, we may choose s_1 with $\deg(s_1^2) \leqslant d+3$ such that $s_1(-1) = a_1$, $s_1(1) = b_1$ and $s_1^2 \leqslant f$, hence $f - s_1^2 \geqslant 0$ on $[-1,1]$. Continuing inductively, we find $s_1, \dots, s_k$ such that $f - \sum_{i=1}^k s_i^2 \geqslant 0$ holds on $[-1,1]$ and $s_i(-1) = a_i$, $s_i(1) = b_i$ for all $i = 1, \dots, k$, with $\deg(s_i^2) \leqslant d + 3i$. By Thm. 2.4, there exist s_{k+1}, s_{k+2} and $r \in \sum \mathbb{R}[t]^2$ such that

$$f - \sum_{i=1}^{k} s_i^2 = s_{k+1}^2 + s_{k+2}^2 + r \cdot (1 - t^2)$$

and with $\deg(s_{k+1}^2), \deg(s_{k+2}^2), \deg(r) \leqslant d + 3k + 1$, which is the desired representation. Note that condition (3) follows automatically from (2), since $s_{k+1}^2(\pm 1) + s_{k+2}^2(\pm 1) = f(\pm 1) - \sum_{i=1}^k s_i^2(\pm 1) = 0$. □

Proof of Thm. 2.6. We do induction on the number e of edges in Δ. If $e = 0$, then Δ is just the set $\{v_1, \dots, v_{m_0}\}$ of isolated vertices. In this case, $T_i = T_i^2$ for all i and $F = \sum_{i=1}^{m_0} F(v_i) T_i$ is a sum of squares.

If Δ is not connected, say $\Delta = \Delta_1 \cup \Delta_2$ with $\Delta_1 \cap \Delta_2 = \emptyset$ and $\Delta_1, \Delta_2 \neq \emptyset$, then we can write $\mathcal{C}^0(\Delta) = \mathcal{C}^0(\Delta_1) \times \mathcal{C}^0(\Delta_2)$. Applying the induction hypothesis to Δ_1 and Δ_2 gives the result.

Now assume that Δ is connected. If $e = 1$, the statement reduces to that of Corollary 2.5, so we assume $e \geqslant 2$ and $(1,2) \in E_\Delta$. Let $\Delta_1 = \Delta(v_1, v_2)$, and let Δ_2 be the closure of $\Delta \setminus \Delta_1$ so that $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2 \subset \{v_1, v_2\}$. We treat the case $\Delta_1 \cap \Delta_2 = \{v_1, v_2\}$. (If $\Delta_1 \cap \Delta_2$ contains only one vertex, the argument is analogous but somewhat simpler.)

Let $F \in \mathcal{C}^0(\Delta)$ be non-negative on Δ. By Prop. 1.4, we can write $F = (f(t), F_2)$ with $f \in \mathbb{R}[t]$ a polynomial in one variable satisfying $f(-1) = F_2(v_1)$ and $f(1) = F_2(v_2)$. By the induction hypothesis applied to F_2, there is a representation

$$F_2 = \sum_{i=1}^{2(e-1)} \widetilde{S}_i^2 + \sum_{(i,j) \in E_{\Delta_2}} \widetilde{R}_{ij} T_i T_j.$$

with $\widetilde{R}_{ij}$ sums of two squares in $\mathcal{C}^0(\Delta_2)$, and with $\deg(\widetilde{S}_i^2), \deg(\widetilde{R}_{ij}) \leqslant \deg(F) + 6(e-2) + 1$. By Cor. 2.9, we can write

$$f = \sum_{i=1}^{2e} s_i^2 + r_{12}(1 - t^2)$$

such that $s_i(-1) = S_i(v_1)$ and $s_i(1) = S_i(v_2)$ for all $i = 1, \dots, 2(e-1)$ and $s_i(-1) = s_i(1) = 0$ for $i = 2e-1, 2e$, where $\deg(s_i^2) \leqslant \deg(f) + 6(e-1) + 1$. It follows that

$$S_i = \begin{cases} (s_i, \widetilde{S}_i) & \text{for } i = 1, \dots, 2(e-1) \\ (s_i, 0) & \text{for } i = 2e-1, 2e \end{cases}$$

are well-defined elements of $C^0(\Delta)$. Choose sums of two squares $R_{ij} \in \sum \mathcal{C}^0(\Delta)$, $(i,j) \in E$, such that $R_{12}|_{\Delta_1} = r_{12}$, $R_{ij}|_{\Delta_2} = \widetilde{R}_{ij}$, and $\deg(R_{12}) = \deg(r_{12})$, $\deg(R_{ij}) = \deg(\widetilde{R}_{ij})$, for all $(i,j) \in E_{\Delta_2}$ (see Remark 1.7). Since T_{ij} is supported on $\Delta(v_i, v_j)$, we see that

$$F = \sum_{i=1}^{2e} S_i^2 + \sum_{(i,j) \in E_\Delta} R_{ij} T_i T_j.$$

□

Examples 2.10. (1) Since the tent functions are themselves non-negative on Δ, they must have a representation as in Theorem 2.6. This is reflected in the simple identity $T_i = T_i(\sum_{j=1}^m T_j) = \sum_{j=1}^m T_i T_j$.

(2) Let Δ be the boundary of the triangle spanned by $v_1 = (0,1)$, $v_2 = (0,0)$, $v_3 = (1,0)$ in $\mathbb{R}^2$. The Zariski-closure of Δ is the plane curve $C = \{(x,y) \in \mathbb{R}^2 \mid xy(1-x-y) = 0\}$, a union of three lines. We can write $\mathcal{C}^0(\Delta)$ in terms of the tent functions T_1, T_2, T_3 with the relations $T_1 + T_2 + T_3 = 1$ and $T_1T_2T_3 = 0$, or we can identify it with the coordinate ring $\mathbb{R}[C]$ which is isomorphic to

$$\big\{(f,g,h) \in \mathbb{R}[u] \times \mathbb{R}[v] \times \mathbb{R}[w] \mid f(1) = g(-1), g(1) = h(-1), h(1) = f(-1)\big\}$$

in such a way that $2T_1 = (u+1, 1-v, 0)$, $2T_2 = (1-u, 0, 1+w)$, $2T_3 = (0, v+1, 1-w)$ (see Prop. 1.4). Consider the function $F = (u^2, v^2, w^2) = 1 - 4T_1T_2 - 4T_1T_3 - 4T_2T_3$. From the second expression, it is not immediately clear that F is non-negative on Δ, while this is obvious from the first, since F is even non-negative on all of C. But F is not a square in $\mathbb{R}[C]$, since $(u,v,w) \notin \mathbb{R}[C]$, nor is it even a sum of squares in $\mathbb{R}[C]$ (see [6], Example (1) in the introduction). However, by Thm. 2.6, it is contained in the quadratic module

$$\mathrm{QM}(T_1T_2, T_1T_3, T_2T_3) = \mathrm{QM}((1-u^2,0,0),(0,1-v^2,0),(0,0,1-w^2)).$$

Using the idea of the proof of Thm. 2.6, one quickly arrives at the representation $F = (u^2, -v, w)^2 + (u, -v, -1)^2(1-u^2, 0, 0)$. Translated into tent functions, this corresponds to the equality

$$1 - 4T_1T_2 - 4T_1T_3 - 4T_2T_3 = (4T_1T_2 - 2T_1 - 2T_2 + 1)^2 + (4T_1 - 2)^2 T_1T_2.$$

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